

TABLE 2.1 Relationships between Rectangular, Cylindrical, and Spherical Coordinates

Rectangular to Cylindrical

$$\text{Variable change} \begin{cases} x = \rho \cos \phi \\ y = \rho \sin \phi \\ z = z \end{cases}$$

$$\text{Component change} \begin{cases} A_\rho = A_x \cos \phi + A_y \sin \phi \\ A_\phi = -A_x \sin \phi + A_y \cos \phi \\ A_z = A_z \end{cases}$$

Cylindrical to Rectangular

$$\text{Variable change} \begin{cases} \rho = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}\left(\frac{y}{x}\right) \\ z = z \end{cases} \begin{cases} \sin \phi = \frac{y}{\sqrt{x^2 + y^2}} \\ \cos \phi = \frac{x}{\sqrt{x^2 + y^2}} \end{cases}$$

$$\text{Component change} \begin{cases} A_x = A_\rho \frac{x}{\sqrt{x^2 + y^2}} - A_\phi \frac{y}{\sqrt{x^2 + y^2}} \\ A_y = A_\rho \frac{y}{\sqrt{x^2 + y^2}} + A_\phi \frac{x}{\sqrt{x^2 + y^2}} \\ A_z = A_z \end{cases}$$

Rectangular to Spherical

$$\text{Variable change} \begin{cases} x = r \sin \theta \cos \phi \\ y = r \sin \theta \sin \phi \\ z = r \cos \theta \end{cases}$$

$$\text{Component change} \begin{cases} A_r = A_x \sin \theta \cos \phi + A_y \sin \theta \sin \phi + A_z \cos \theta \\ A_\theta = A_x \cos \theta \cos \phi + A_y \cos \theta \sin \phi - A_z \sin \theta \\ A_\phi = -A_x \sin \phi + A_y \cos \phi \end{cases}$$

Spherical to Rectangular

$$\text{Variable change} \begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = \cos^{-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}} \\ \phi = \tan^{-1}\left(\frac{y}{x}\right) \end{cases} \begin{cases} \cos \theta = \frac{z}{\sqrt{x^2 + y^2 + z^2}} \\ \sin \theta = \frac{\sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}} \\ \begin{cases} \cos \phi = \frac{x}{\sqrt{x^2 + y^2}} \\ \sin \phi = \frac{y}{\sqrt{x^2 + y^2}} \end{cases} \end{cases}$$

$$\text{Component change} \begin{cases} A_x = \frac{A_r x}{\sqrt{x^2 + y^2 + z^2}} + \frac{A_\theta x z}{\sqrt{(x^2 + y^2)(x^2 + y^2 + z^2)}} - \frac{A_\phi y}{\sqrt{x^2 + y^2}} \\ A_y = \frac{A_r y}{\sqrt{x^2 + y^2 + z^2}} + \frac{A_\theta y z}{\sqrt{(x^2 + y^2)(x^2 + y^2 + z^2)}} + \frac{A_\phi x}{\sqrt{x^2 + y^2}} \\ A_z = \frac{A_r z}{\sqrt{x^2 + y^2 + z^2}} - \frac{A_\theta \sqrt{x^2 + y^2}}{\sqrt{x^2 + y^2 + z^2}} \end{cases}$$

Adopted with permission from G. E. Miner, *Lines and Electromagnetic Fields for Engineers*. New York: Oxford Univ. Press, 1996, p. 263.

REVIEW QUESTIONS

2.1 The ranges of θ and ϕ as given by eq. (2.17) are not the only possible ones. The following are all alternative ranges of θ and ϕ , except

- (a) $0 \leq \theta < 2\pi, 0 \leq \phi \leq \pi$
- (b) $0 \leq \theta < 2\pi, 0 \leq \phi < 2\pi$
- (c) $-\pi \leq \theta \leq \pi, 0 \leq \phi \leq \pi$
- (d) $-\pi/2 \leq \theta \leq \pi/2, 0 \leq \phi < 2\pi$
- (e) $0 \leq \theta \leq \pi, -\pi \leq \phi < \pi$
- (f) $-\pi \leq \theta < \pi, -\pi \leq \phi < \pi$

2.2 At Cartesian point $(-3, 4, -1)$, which of these is incorrect?

- | | |
|---------------------|---------------------------------------|
| (a) $\rho = -5$ | (c) $\theta = \tan^{-1} \frac{5}{-1}$ |
| (b) $r = \sqrt{26}$ | (d) $\phi = \tan^{-1} \frac{4}{-3}$ |

2.3 Which of these is not valid at point $(0, 4, 0)$?

- | | |
|---|--------------------------------------|
| (a) $\mathbf{a}_\phi = -\mathbf{a}_x$ | (c) $\mathbf{a}_r = 4\mathbf{a}_y$ |
| (b) $\mathbf{a}_\theta = -\mathbf{a}_z$ | (d) $\mathbf{a}_\rho = \mathbf{a}_y$ |

2.4 A unit normal vector to the cone $\theta = 30^\circ$ is:

- | | |
|-------------------------|-----------------------|
| (a) $\mathbf{a}_r$ | (c) $\mathbf{a}_\phi$ |
| (b) $\mathbf{a}_\theta$ | (d) none of these |

2.5 At every point in space, $\mathbf{a}_\phi \cdot \mathbf{a}_\theta = 1$.

- | | |
|----------|-----------|
| (a) True | (b) False |
|----------|-----------|

2.6 If $\mathbf{H} = 4\mathbf{a}_\rho - 3\mathbf{a}_\phi + 5\mathbf{a}_z$, at $(1, \pi/2, 0)$ the component of $\mathbf{H}$ parallel to surface $\rho = 1$ is

- | | |
|-------------------------|---|
| (a) $4\mathbf{a}_\rho$ | (d) $-3\mathbf{a}_\phi + 5\mathbf{a}_z$ |
| (b) $5\mathbf{a}_z$ | (e) $5\mathbf{a}_\phi + 3\mathbf{a}_z$ |
| (c) $-3\mathbf{a}_\phi$ | |

2.7 Given $\mathbf{G} = 20\mathbf{a}_r + 50\mathbf{a}_\theta + 40\mathbf{a}_\phi$, at $(1, \pi/2, \pi/6)$ the component of $\mathbf{G}$ perpendicular to surface $\theta = \pi/2$ is

- | | |
|---------------------------|--|
| (a) $20\mathbf{a}_r$ | (d) $20\mathbf{a}_r + 40\mathbf{a}_\theta$ |
| (b) $50\mathbf{a}_\theta$ | (e) $-40\mathbf{a}_r + 20\mathbf{a}_\phi$ |
| (c) $40\mathbf{a}_\phi$ | |

2.8 Where surfaces $\rho = 2$ and $z = 1$ intersect is

- | | |
|---------------------------|----------------|
| (a) an infinite plane | (d) a cylinder |
| (b) a semi-infinite plane | (e) a cone |
| (c) a circle | |

2.9 Match the items in the list at the left with those in the list at the right. Each answer can be used once, more than once, or not at all.

- | | |
|---|--------------------------|
| (a) $\theta = \pi/4$ | (i) infinite plane |
| (b) $\phi = 2\pi/3$ | (ii) semi-infinite plane |
| (c) $x = -10$ | (iii) circle |
| (d) $r = 1, \theta = \pi/3, \phi = \pi/2$ | (iv) semicircle |
| (e) $\rho = 5$ | (v) straight line |
| (f) $\rho = 3, \phi = 5\pi/3$ | (vi) cone |
| (g) $\rho = 10, z = 1$ | (vii) cylinder |

- (h) $r = 4, \phi = \pi/6$ (viii) sphere
 (i) $r = 5, \theta = \pi/3$ (ix) cube
 (x) point

2.10 A wedge is described by $z = 0, 30^\circ < \phi < 60^\circ$. Which of the following is incorrect?

- (a) The wedge lies in the xy -plane.
 (b) It is infinitely long.
 (c) On the wedge, $0 < \rho < \infty$.
 (d) A unit normal to the wedge is $\pm \mathbf{a}_z$.
 (e) The wedge includes neither the x -axis nor the y -axis.

Answers: 2.1b,f, 2.2a, 2.3c, 2.4b, 2.5b, 2.6d, 2.7b, 2.8c, 2.9a-(vi), b-(ii), c-(i), d-(x), e-(vii), f-(v), g-(iii), h-(iv), i-(iii), 2.10b.

PROBLEMS

Sections 2.3 and 2.4—Cylindrical and Spherical Coordinates

- 2.1** The rectangular coordinates at point P are $(x = 2, y = 6, z = -4)$. (a) What are its cylindrical coordinates? (b) What are its spherical coordinates?
- 2.2** Express the following points in cylindrical and spherical coordinates:
- (a) $P(1, -4, -3)$
 (b) $Q(3, 0, 5)$
 (c) $R(-2, 6, 0)$
- 2.3** Express the following points in Cartesian coordinates:
- (a) $P_1(2, 30^\circ, 5)$
 (b) $P_2(1, 90^\circ, -3)$
 (c) $P_3(10, \pi/4, \pi/3)$
 (d) $P_4(4, 30^\circ, 60^\circ)$
- 2.4** The cylindrical coordinates of point Q are $\rho = 5, \phi = 120^\circ, z = 1$. Express Q as rectangular and spherical coordinates.
- 2.5** Given point $T(10, 60^\circ, 30^\circ)$ in spherical coordinates, express T in Cartesian and cylindrical coordinates.
- 2.6** (a) If $V = xz - xy + yz$, express V in cylindrical coordinates.
 (b) If $U = x^2 + 2y^2 + 3z^2$, express U in spherical coordinates.
- 2.7** Convert the following vectors to cylindrical and spherical systems:

(a) $\mathbf{F} = \frac{x\mathbf{a}_x + y\mathbf{a}_y + 4\mathbf{a}_z}{\sqrt{x^2 + y^2 + z^2}}$

$$(b) \mathbf{G} = (x^2 + y^2) \left[\frac{x\mathbf{a}_x}{\sqrt{x^2 + y^2 + z^2}} + \frac{y\mathbf{a}_y}{\sqrt{x^2 + y^2 + z^2}} + \frac{z\mathbf{a}_z}{\sqrt{x^2 + y^2 + z^2}} \right]$$

2.8 Let $\mathbf{B} = \sqrt{x^2 + y^2} \mathbf{a}_x + \frac{y}{\sqrt{x^2 + y^2}} \mathbf{a}_y + z\mathbf{a}_z$. Transform $\mathbf{B}$ to cylindrical coordinates.

2.9 Given vector $\mathbf{A} = 2\mathbf{a}_\rho + 3\mathbf{a}_\phi + 4\mathbf{a}_z$, convert $\mathbf{A}$ into Cartesian coordinates at point $(2, \pi/2, -1)$.

2.10 Express the following vectors in rectangular coordinates:

$$(a) \mathbf{A} = \rho \sin \phi \mathbf{a}_\rho + \rho \cos \phi \mathbf{a}_\phi - 2z \mathbf{a}_z$$

$$(b) \mathbf{B} = 4r \cos \phi \mathbf{a}_r + r \mathbf{a}_\theta$$

2.11 Express vector $\mathbf{G} = \rho \sin \phi \mathbf{a}_\rho - \rho \cos \phi \mathbf{a}_\phi + \rho \mathbf{a}_z$ in rectangular coordinates.

2.12 Express vector $\mathbf{H} = \cos \theta \mathbf{a}_r + \sin \theta \mathbf{a}_\theta$ in rectangular coordinates.

2.13 Let $\mathbf{B} = x\mathbf{a}_z$. Express $\mathbf{B}$ in

(a) cylindrical coordinates,

(b) spherical coordinates.

2.14 Prove the following:

$$(a) \mathbf{a}_x \cdot \mathbf{a}_\rho = \cos \phi$$

$$\mathbf{a}_x \cdot \mathbf{a}_\phi = -\sin \phi$$

$$\mathbf{a}_y \cdot \mathbf{a}_\rho = \sin \phi$$

$$\mathbf{a}_y \cdot \mathbf{a}_\phi = \cos \phi$$

$$(b) \mathbf{a}_x \cdot \mathbf{a}_r = \sin \theta \cos \phi$$

$$\mathbf{a}_x \cdot \mathbf{a}_\theta = \cos \theta \cos \phi$$

$$\mathbf{a}_y \cdot \mathbf{a}_r = \sin \theta \sin \phi$$

$$(c) \mathbf{a}_y \cdot \mathbf{a}_\theta = \cos \theta \sin \phi$$

$$\mathbf{a}_z \cdot \mathbf{a}_r = \cos \theta$$

$$\mathbf{a}_z \cdot \mathbf{a}_\theta = -\sin \theta$$

2.15 (a) Show that point transformation between cylindrical and spherical coordinates is obtained using

$$r = \sqrt{\rho^2 + z^2}, \quad \theta = \tan^{-1} \frac{\rho}{z}, \quad \phi = \phi$$

or

$$\rho = r \sin \theta, \quad z = r \cos \theta, \quad \phi = \phi$$

(b) Show that vector transformation between cylindrical and spherical coordinates is obtained using

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin \theta & 0 & \cos \theta \\ \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

or

$$\begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \\ \cos \theta & -\sin \theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

(Hint: Make use of Figures 2.5 and 2.6.)

2.16 Show that the vector fields

$$\mathbf{A} = \rho \sin \phi \mathbf{a}_\rho + \rho \cos \phi \mathbf{a}_\phi + \rho \mathbf{a}_z$$

$$\mathbf{B} = \rho \sin \phi \mathbf{a}_\rho + \rho \cos \phi \mathbf{a}_\phi - \rho \mathbf{a}_z$$

are perpendicular to each other at any point.

2.17 Given that $\mathbf{A} = 3\mathbf{a}_\rho + 2\mathbf{a}_\phi + \mathbf{a}_z$ and $\mathbf{B} = 5\mathbf{a}_\rho - 8\mathbf{a}_z$, find:(a) $\mathbf{A} + \mathbf{B}$, (b) $\mathbf{A} \cdot \mathbf{B}$, (c) $\mathbf{A} \times \mathbf{B}$, (d) the angle between $\mathbf{A}$ and $\mathbf{B}$.2.18 Let $\mathbf{A} = \rho \cos \phi \mathbf{a}_\rho + \rho z^2 \sin \phi \mathbf{a}_z$.(a) Transform $\mathbf{A}$ into rectangular coordinates and calculate its magnitude at point $(3, -4, 0)$.(b) Transform $\mathbf{A}$ into spherical system and calculate its magnitude at point $(3, -4, 0)$.2.19 The transformation $(A_\rho, A_\phi, A_z) \rightarrow (A_x, A_y, A_z)$ in eq. (2.15) is not complete. Complete it by expressing $\cos \phi$ and $\sin \phi$ in terms of x, y , and z . Do the same thing to the transformation $(A_r, A_\theta, A_\phi) \rightarrow (A_x, A_y, A_z)$ in eq. (2.28).2.20 In Practice Exercise 2.2, express $\mathbf{A}$ in spherical and $\mathbf{B}$ in cylindrical coordinates. Evaluate $\mathbf{A}$ at $(10, \pi/2, 3\pi/4)$ and $\mathbf{B}$ at $(2, \pi/6, 1)$.

2.21 Calculate the distance between the following pairs of points:

(a) $(2, 1, 5)$ and $(6, -1, 2)$ (b) $(3, \pi/2, -1)$ and $(5, 3\pi/2, 5)$ (c) $(10, \pi/4, 3\pi/4)$ and $(5, \pi/6, 7\pi/4)$ 2.22 Given points $P(10, \pi/4, 0)$ and $Q(4, \pi/2, \pi/2)$, find the distance between P and Q .

2.23 Describe the intersection of the following surfaces:

(a) $x = 2, y = 5$ (b) $x = 2, y = -1, z = 10$ (c) $r = 10, \theta = 30^\circ$ (d) $\rho = 5, \phi = 40^\circ$ (e) $\phi = 60^\circ, z = 10$ (f) $r = 5, \phi = 90^\circ$

2.24 Given vectors $\mathbf{A} = 2\mathbf{a}_x + 4\mathbf{a}_y + 10\mathbf{a}_z$ and $\mathbf{B} = -5\mathbf{a}_\rho + \mathbf{a}_\phi - 3\mathbf{a}_z$, find

- (a) $\mathbf{A} + \mathbf{B}$ at $P(0, 2, -5)$
- (b) The angle between $\mathbf{A}$ and $\mathbf{B}$ at P
- (c) The scalar component of $\mathbf{A}$ along $\mathbf{B}$ at P

2.25 Let $\mathbf{A} = (2z - \sin \phi)\mathbf{a}_\rho + (4\rho + 2\cos \phi)\mathbf{a}_\phi - 3\rho z\mathbf{a}_z$ and $\mathbf{B} = \rho \cos \phi \mathbf{a}_\rho + \sin \phi \mathbf{a}_\phi + \mathbf{a}_z$.

- (a) Find the minimum angle between $\mathbf{A}$ and $\mathbf{B}$ at $(1, 60^\circ, -1)$.
- (b) Determine a unit vector normal to both $\mathbf{A}$ and $\mathbf{B}$ at $(1, 90^\circ, 0)$.

2.26 At point $T(2, 3, -4)$, express $\mathbf{a}_z$ in the spherical system and $\mathbf{a}_r$ in the rectangular system.

2.27 Given that $\mathbf{G} = 6r^2 \sin \phi \mathbf{a}_r + r^2 \mathbf{a}_\phi$, find $\mathbf{G} \cdot \mathbf{a}_y$ at $(2, -3, 1)$.

2.28 A vector field in "mixed" coordinate variables is given by

$$\mathbf{G} = \frac{x \cos \phi}{\rho} \mathbf{a}_x + \frac{2yz}{\rho^2} \mathbf{a}_y + \left(1 - \frac{x^2}{\rho^2}\right) \mathbf{a}_z$$

Express $\mathbf{G}$ completely in the spherical system.

Section 2.5—Constant-Coordinate Surfaces

2.29 If $\mathbf{J} = r \sin \theta \cos \phi \mathbf{a}_r - \cos 2\theta \sin \phi \mathbf{a}_\theta + \tan \frac{\theta}{2} \ln r \mathbf{a}_\phi$ at $T(2, \pi/2, 3\pi/2)$, determine the vector component of $\mathbf{J}$ that is:

- (a) Parallel to $\mathbf{a}_z$
- (b) Normal to surface $\phi = 3\pi/2$
- (c) Tangential to the spherical surface $r = 2$
- (d) Parallel to the line $y = -2, z = 0$

2.30 If $\mathbf{H} = \rho^2 \cos \phi \mathbf{a}_\rho - \rho \sin \phi \mathbf{a}_\phi$, find $\mathbf{H} \cdot \mathbf{a}_x$ at point $P(2, 60^\circ, -1)$.

2.31 If $\mathbf{r} = x\mathbf{a}_x + y\mathbf{a}_y + z\mathbf{a}_z$, describe the surface defined by:

- (a) $\mathbf{r} \cdot \mathbf{a}_x + \mathbf{r} \cdot \mathbf{a}_y = 5$
- (b) $|\mathbf{r} \times \mathbf{a}_z| = 10$